Name of college -

Dept -

Topic - Solved Example on De Morgan's or Bertrand's Test (Infinite series)

Class - B.Sc II

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Problem 1

Test the series

$$\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2}x + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2 + \cdots$$

Sol^n → Leaving the first term, if the given series be Σu_n

If u_n be the nth term of the series then $u_n = \frac{1^2.3^2 \cdots (2n+1)^2}{2^2.4^2 \cdots (2n+2)^2}x^n$

and hence $u_{n+1} = \frac{1^2.3^2 \cdots (2n+1)^2 (2n+3)^2}{2^2.4^2 \cdots (2n+2)^2 (2n+4)^2}x^{n+1}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+4)^2}{(2n+3)^2} \cdot \frac{1}{x}$$

$$= \frac{4n^2+16n+16}{4n^2+12n+9} \cdot \frac{1}{x}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{4n^2+16n+16}{4n^2+12n+9} \frac{1}{x} = \frac{1}{x} \quad \text{--- (I)}$$

Therefore, from ~~Ratio~~ Ratio Test :-

Σu_n is convergent if $\frac{1}{x} > 1$
$\Rightarrow x < 1$

and divergent if $\frac{1}{x} < 1$
$\Rightarrow x > 1$ — (II)

If $\frac{1}{x} = 1 \Rightarrow$ Ratio Test Fails
and hence from (1) we have

$$\frac{u_n}{u_{n+1}} = \frac{4n^2+16n+16}{4n^2+12n+9}$$

& $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{4n^2+16n+16}{4n^2+12n+9} - 1\right]$$

$$= n\left[\frac{4n+7}{4n^2+12n+9}\right] \quad \text{— (III)}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n+7}{4n^2+12n+9}\right)$$

$$= \lim_{n\to\infty} \frac{n^2(4+7/n)}{n^2\left(4+\frac{12}{n}+\frac{9}{n^2}\right)}$$

$\therefore$ Raabe's Test Fails and

Therefore, From ~~Ratio~~ Ratio Test :-

Σu_n is convergent if $\frac{1}{x} > 1$
$\Rightarrow x < 1$

and divergent if $\frac{1}{x} < 1$
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If $\frac{1}{x} = 1 \Rightarrow$ Ratio Test Fails
and hence From (I) we have

$$\frac{u_n}{u_{n+1}} = \frac{4n^2 + 16n + 16}{4n^2 + 12n + 9}$$

& $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{4n^2 + 16n + 16}{4n^2 + 12n + 9} - 1\right]$$

$$= n\left[\frac{4n + 7}{4n^2 + 12n + 9}\right] \text{ — (III)}$$

$$\therefore \lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n \to \infty} n\left(\frac{4n + 7}{4n^2 + 12n + 9}\right)$$

$$= \lim_{n \to \infty} \frac{n^2(4 + 7/n)}{n^2\left(4 + \frac{12}{n} + \frac{9}{n^2}\right)}$$ —

$\therefore$ Raabe's Test Fails and

Hence from (iii) we have

$$n\left(\frac{u_n}{u_{n+1}}-1\right)-1 = \frac{4n^2+7n}{4n^2+12n+9}-1 = \frac{-5n-9}{4n^2+12n+9}$$

or $$\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = -\frac{(5n+9)}{4n^2+12n+9}\log n$$

$$= -\frac{n(5+9/n)}{n^2\left[4+\frac{12}{n}+\frac{9}{n^2}\right]}\log n$$

$$= -\frac{(5+9/n)}{\left(4+\frac{12}{n}+\frac{9}{n^2}\right)}\frac{\log n}{n}.$$

& $$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n$$

$$= \lim_{n\to\infty} -\frac{(5+9/n)}{\left(4+\frac{12}{n}+\frac{9}{n^2}\right)}\frac{\log n}{n}$$

$$= -5/4 \cdot 0 \qquad \because \lim_{n\to\infty}\frac{\log n}{n}=0$$

$$= 0 < 1$$

Hence from De Morgan's Test.
The given series is divergent.
Thus we see the series is convergent if $x < 1$
and divergent if $x \geq 1$

Ex. 2 Test the convergency the series

$$\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \cdots$$

Solution – If Σu_n be the given series

then $$u_n = \frac{a(a+1)(a+2)\cdots(a+n-1)}{b(b+1)(b+2)\cdots(b+n-1)}$$

$$\therefore u_{n+1} = \frac{a(a+1)(a+2)\cdots(a+n-1)}{b(b+1)(b+2)\cdots(b+n-1)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{b+n}{a+n} = \frac{n(b/n+1)}{n(a/n+1)} \quad \text{—} (1)$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{b/n+1}{a/n+1}$$

$$= 1$$

Ratio Test fails and from (1) we have

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{n+b}{n+a} - 1\right) = \frac{n(b-a)}{n+a} \quad \text{—} (ii)$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\left[\frac{n(b-a)}{n+a}\right]$$

$$= \lim_{n\to\infty} \frac{b-a}{1+a/n}$$

$$= b-a$$

∴ From Raabe's Test we find that Given series is convergent if

$$b-a>1$$

and Divergent if $b-a<1$

i.e. $b > a+1$ (convergent)

$a < a+1$ (Divergent)

If $b-a=1$, Raabe's Test Fails and hence From (4)

we have

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{n+a}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{n}{n+a} - 1 = \frac{-a}{n+a}$$

$$\Rightarrow \left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right]\log n = -\frac{a\log n}{n+a} = -\frac{a}{1+a/n}\frac{\log n}{n}$$

$\therefore \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right]\log n$

$$=\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right]\log n$$

$$=\lim_{n\to\infty} -\frac{a}{1+a/n}\cdot\frac{\log n}{n}$$

$$=0<1$$

Hence from De-Morgan's Test.
the Given Series is Divergent
if $b-a=1$
i-e $b=a+1$

Thus we find here.
The Given Series is Convergent
if $b>1+a$ and
divergent if $b<1+a$.